Perspective and Pictorial Space

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1 Linear perspective

As discussed by, for example, White in “The Birth and Rebirth of Pictorial Space” [1] and Kemp in “The Science of Art” [2], the illusionist technique of linear perspective was developed by Brunelleschi 1 and described by Alberti in a Latin manuscript dated 1440; a modern translation is available [3]. As discussed by Kemp [2], the geometry of perspective was developed over the centuries to give a comprehensive body of work that artists could use to construct a pictorial space according to a set of strict rules. These geometrical rules become quite complicated for curved objects and so only the simplest situations are described here.

The basic principles of perspective appear in Figure 1 2. Straight rays emanating from each point (A, B, C, D, and E)) of the scene impinge on the observers eye located at O. The rays also pass through the picture plane FGH; thus, if a picture is correctly constructed on the picture

![Figure 1: Principles of Perspective](image)
plane, the rays at the observer’s eye emanating from the picture plane points \(a, b, c, d,\) and \(e\) would appear to come from the real points \(A, B, C, D,\) and \(E.\) All of the points appearing in the picture plane thus form a two-dimensional image of the three-dimensional scene.\(^3\)

It is important to realise that Figure 1 not only shows how to construct the image on the picture plane \(FGHI\) but also how to view the image so that it appears to be the actual scene. In particular, the eye location \(O\) must be the same (relative to the picture plane) for both creating and viewing the image.\(^4\) The simplest case arises when the ray through the centre of the frame is perpendicular to the picture plane; this gives an horizon line half way up the picture and a viewing point on that ray a distance \(\Delta\) from the picture plane. The horizontal Field of View (FoV) is then the angle subtended at the eye by this horizon line. In photographers jargon, an image with a large FoV (say greater than \(70^\circ\)) is a wide-angle image and an image with a small FoV (say less than \(40^\circ\)) is a narrow-angle image. In this special case, the result of the perspective construction is illustrated in Figure 2 for two images: a computer-generated tiled floor and a landscape photograph of Loch Ossian. If the width of the picture is 1m, the distance \(\Delta\) of the picture plane to the viewer is as given in the third column of Table 1; the corresponding focal length for a camera with a sensor width of 36mm is given in the forth column.\(^5\) Some consequences are:

1. If a picture is reproduced at half the size, the correct viewing distance \(\Delta\) is halved, and vice versa. It follows that wide-angle images need to be large to achieve a comfortable viewing distance.\(^7\)

2. Given a scene and a viewing point, and two canvases of the same size at distances \(\Delta_1\) and \(\Delta_2\) with \(\Delta_2\) greater than \(\Delta_1,\) the canvas at \(\Delta_2\) can be reproduced at a smaller size to exactly fit the central part of the image on the canvas at \(\Delta_1\).\(^8\)

The same results hold for photographs where \(\Delta_1\) and \(\Delta_2\) are the corresponding focal lengths.

Figure 3(a) shows an example of a wide-angle perspectival painting - a large mural in the Vatican. The tiles correspond approximately to those of Figure 2(e). On the other hand, Figure 3(b) shows a narrow-angle perspectival painting; a smaller canvas in the National Gallery of Victoria. The tiles correspond approximately to those of Figure 2(a). In the former picture, the horizon line is higher than the centre giving an illusion of looking down; the reverse is true for the latter picture.

As illustrated in Figure 4, there are three ways of creating a perspectival image:

<table>
<thead>
<tr>
<th>Figs.</th>
<th>Field of view (FoV)</th>
<th>Distance (\Delta) (m)</th>
<th>Focal length (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a &amp; b</td>
<td>22.5°</td>
<td>2.5</td>
<td>90</td>
</tr>
<tr>
<td>c &amp; d</td>
<td>45°</td>
<td>1.2</td>
<td>43</td>
</tr>
<tr>
<td>e &amp; f</td>
<td>90° (^5)</td>
<td>0.5</td>
<td>18</td>
</tr>
<tr>
<td>g &amp; h</td>
<td>135°</td>
<td>0.2</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 1: Values for Figure 2
Geometrical construction: Figure 4(a) shows Botticelli’s *The Annunciation* (c.1480) with construction lines added. This illustrates the most basic procedure of geometrical construction: parallel straight lines in the scene orthogonal to the picture plane become converging straight lines in the image; the point of convergence is called the vanishing point. The description of this picture (located in Glasgow’s Kelvingrove gallery) notes that construction markings are still visible on close inspection of this painting.

Mechanical means: Figure 4(b) is a woodcut by Albrecht Dürer dated 1525 which shows a mechanical way of plotting the key points of an image. In essence, the eye is replaced by a pulley and each light ray of Figure 1 is replaced consecutively by a string stretched between the pulley and a point on the scene. The point where the string passes through the picture plane is recorded by the second operator using a hinged drawing surface. There are many such machines of varying sophistication described by Kemp [2].

Optical means: Figure 4(c) is a woodcut (provenance unknown) describing a simple *camera obscura* (literally a dark room). Light though a pinhole illuminates a translucent screen to give an inverted image of the scene. The lays of optics make this equivalent to the procedure of Figure 1. Again, Kemp [2] describes a sequence of such machines developed though the centuries; the “optical telescope” (1811) of Cornelius Varley [2, pll. 403–405] being a particularly fine example. According to Kemp [2], a number of artists, including Vermeer and Canaletto, made use of the *camera obscura*.

The modern digital single-lens reflex camera is the culmination of this optical development.

All of the above procedures are subject to error of one form or another; but, disregarding such errors, they would all produce exactly the same image given the same viewpoint and equivalent picture plane. For this reason, the truth or otherwise of the Hockney/Falco hypothesis, is not relevant to this paper.

As noted previously in relation to Figure 1, there is a key issue related to linear perspective: for each particular scene, the image on the picture plane depends not only on the location of the eye but also on the the position and orientation of the picture plane. Following Figure 1, this means not only that

1. the image is to be viewed from *exactly the same point* relative to the image as the eye point used in its creation but also that

2. the image orientation is exactly the same.

The first point is illustrated by Figure 5(a), *The New Town Hall in Amsterdam* by van der Heyden (1667). The Dome, which is hemispherical in reality, appears distorted when looked at from normal viewing distance at the size that it is reproduced in the figure; detail of the dome appears in Figure 6(a). In fact, according to Kemp [2], p. 206 the viewing instructions included: “the painting should be viewed from its proper point by means of an instrument of iron which is attached to the frame”. Kemp goes on to say that “This instrument probably resembled the
eyepiece of a perspective machine”. To test this, I took a photograph of the pictured dome (displayed on a computer monitor) from an angle I judged to be the correct viewing angle; this appears as Fig. 6(b) – the dome now appears to be hemispherical unlike that in Figure 6(a). Of course such apparent distortion would be less apparent in a landscape not containing precise geometrical shapes.

Exactly the same effect arises in photography. Fig. 7 shows two wide angle photographs; the spherical lanterns in Fig. 7(a) appear distorted towards the edge of the photograph. The same distortion occurs in Fig. 7(b), but it is not apparent as the distorted clouds look entirely natural.

The second point is illustrated by Fig. 8. The image in Fig. 8(a) contains an object at the bottom centre; it is, in fact, a skull painted on an oblique surface which has been added to the “normal” picture. Fig. 8(b) shows Fig. 8(a) rotated about two axes; it clearly reveals the skull.

Ultimately, this problem arises because images created using the strict rules corresponding to Fig. 1 are intended as the raw material for a peepshow; they are not designed to be hung on a wall and viewed from an arbitrary distance and angle. In fact, as pointed out by Pirenne [4], it is surprising that hanging such pictures on the wall works at all; it is only in cases of extremely wide angle pictures that apparent distortion occurs. This effect is, as discussed by Pirenne [4], Kubovy [5] and Gombrich [6], has more to do with psychology than mathematics. In particular, it seems that the human mind effortlessly removes the distortions caused by an oblique image as long as the angles involved are not too large; on the other hand an image of an oblique image is not so translated – see Figs. 6(b) and 8(b) for two examples of the latter. For this reason artists often break the strict rules of perspective to avoid such apparent distortion at wide angles; this will be called compound perspective in this paper.

### Compound Perspective

The problems of linear perspective can be partially avoided by loosening the strict requirements of overall linear perspectival structure in favour of making local changes where distortions would otherwise be apparent. Such distortions are more apparent when the objects have a known shape: figures and simple geometric solids are the main problems. Moreover, from a purely practical point of view, it would be difficult to make every object in a painting conform exactly to overall perspectival requirements; it would be simpler to paint individual objects in their own local perspectival space. There are many examples of this:

- The figures at the far left and right of the “School at Athens” (Figure 3(a)) are not distorted, and the spheres at the right-hand side appear circular, even though they are at the extremities of a wide-angle picture: see Kubovy [5], Chap. 7, The bounds of Perspective: marginal distortion.

- The figures in Figure 4(a) appear to be painted as actors onto the perspectival scenery.

- In Fig. 5(b), the artist has replaced the dome (Figure 6(a)) of Figure 5(a) with the image of Figure 6(c) taken with a rotated picture plane. This removes the apparent distortion at
the expense of spoiling the illusion when viewed from the original viewpoint. Figure 6(d) a copy of 6(a) with a digitally rotated picture plane; this confirms that the artist has used a rotated picture plane.

- As discussed by Hockney [7], a number of the items shown on the table in Holbein’s “The Ambassadors” (Figure 8) do not conform to the overall perspectival structure of the picture.

The wide angle problem, and its ad hoc solutions, lead to some artists discarding linear perspective altogether. A notable pioneer was Cézanne, who painted a series of still lifes with clearly multiple viewpoints.

Indeed, this movement away from illusionist pictures has been the hallmark of 20th century art. This is a large topic and only one aspect of this, the photo-collages of David Hockney, is noted here. The book of Hockney’s photo-collages [8] contains many example of collages made from photographs; two of which appear in Figure 9. As a camera is an perspectival machine, each photo is individually perspectivally correct; but each photo is taken with a different viewpoint and picture plane. So the overall collage does not conform to linear perspective. Indeed, Hockney describes this as: “a panoramic assault on Renaissance one-point perspective” in [9] and notes that “Multiple viewpoints create a far bigger space than can be achieved by one” in [7]. He goes on to suggest that Renaissance painters such as van Eyck used such an approach.

Discarding the framework of single viewpoint perspective certainly removes one constraint from the creation of wide-angle images. However, is does leave the problem of how to joint the images from the multiple view points together. Cézanne and van Eyck manage to do this in a seamless fashion; Hockney leaves the edges showing.

There is another solution to the wide-angle problem that avoids the joining problem: spherical perspective.

**Spherical perspective**

Although the diagram of Fig. 1 shows a flat picture plane; this is not necessary – for example, Fig. 10 shows a scene painted on to the curved surface of a church ceiling. In particular, the flat picture plane of Fig. 1 can be replaced by a sphere centred on the eye at $O$ – this will be called the viewsphere. This approach has a number of advantages:

1. objects with a given angular size are the same size on the sphere
2. the scene can encompass a horizontal field of view of $360^\circ$ and a vertical field of view of $180^\circ$.

This idea is discussed in a photographic context by German et al. [10] and Gawthrop [11].

There remains the problem of how to “flatten” the viewsphere onto a plane surface. As discussed by Snyder in his book *Flattening the Earth: Two Thousand Years of Map Projections* [12], this problem has been addressed by cartographers since antiquity. And, like perspective itself, the Renaissance saw a renewed interest in the topic. Indeed, at least two Renaissance artists: Dürer and Leonardo made contributions to cartography (Figure 11) and were thus aware
of the issues. White [1], Chapter XIV makes a strong case for the development of a spherically based perspective by Leonardo in a now lost manuscript.

According to Snyder [12], Chap 1, one of the earliest world maps was based on the Equirectangular, or *plate carrée*, projection where lines of latitude and longitude are equally spaced on a map with a 2:1 width:height ratio. Such a map has the advantage of showing the entire surface of the globe; in pictorial terms it can thus represent a horizontal field of view of 360° and a vertical field of view of 180°. Fig. 12(a) shows the equirectangular projection of the tiled floor of Figure 2 and Fig. 12(b) shows the equirectangular projection of the Loch Ossian scene of Figure 2. In each case the full scene (horizontal field of view of 360° and vertical field of view of 180°) is shown. Fig. 12(a) illustrates the difficulty known to Leonardo and explained by White [1], Chapter XIV: straight lines become curved. However, as Fig. 12(b) shows, this is not a such problem for landscapes as it is for architectural scenes.

In 1569, the mathematician and cartographer Gerhardus Mercator presented his celebrated map projection. As illustrated in Figs. 12(c) and 12(d) this expands the upper and lower parts of each image at the expense of restricting the vertical field of view.

Dürer used the Stereographic projection in his star maps of 1515. This projection is also of pictorial interest; Figs. 12(e) & 12(g) show two possibilities for the tiles scene and Figs. 12(f) & 12(h) the corresponding projections for the Loch Ossian scene.

But did Renaissance painters actually use such projections? I have located two paintings which may use wide-angle equirectangular or Mercator projections. Fig. 13(a) shows an illustration from the manuscript *Grandes Chroniques de France, enluminés par Jean Fouquet, Tours, vers 1455-1460*. Fig. 13(b) is a Dutch painting from almost 200 years later.

It is not known how these pictures were produced. I would guess that Fig. 13(a) was drawn with the aid of a convex mirror, despite the fact that such devices were virtually banned by the Pope. Kemp [2] suggests that Fig. 13(b) was produced with the aid of device of Lanci [2, pl.337]. In this context, the widespread use in the 18th century of the “Claude Glass” – a black convex mirror – to view and draw landscapes should be noted [13].

As discussed by German et al. [10] and Gawthrop [11], the availability of digital cameras and appropriate software has lead to new possibilities in using photography to create aesthetically-pleasing projections of spherical panoramas such as those shown in Figures 12(f), 12(d) and 12(h). I expect that over the next few years, these ideas will feed into painting as well thus giving another rebirth of pictorial space.
Notes

1 According to a number of authors including White [1], chapter XVI and Gombrich [6], chapter IV, the delineation of pictorial space dates back at least to classical Greek painting.

2 Figure 1 is Figure 7.1 of reference [4] and is attributed to Brook Taylor (1811) New Principles of Perspective.

3 The method gives the location but not the texture, colour or intensity of each point of the image. These later attributes are equally important to a successful image but are outside the scope of this paper.

4 It is surprising how many pictures are incorrectly hung. In the Kelvingrove Art Gallery, “An Imaginary Church Interior” (1645) by Bartholomeus van Bassen is hung so low that one has to kneel on the floor for the correct viewing position; the same problem occurs for two pictures in the Hunterian Art Gallery: “A Sacrifice to Apollo” (1740) by John Devoto and “Architectural Fantasy” (1630) by Paul Vredeman de Vries.

5 This is a special case. As discussed by [2], the vanishing points corresponding to the tile diagonals are separated by the viewing distance \( \Delta \); so, in this case, they are at the edge of the picture.

6 Defining \( \Delta \) as the distance to the picture from the eye, \( W \) the width of the picture frame and \( \alpha \) the field of view then the relevant formula is

\[
\alpha = 2 \tan \left( \frac{W}{2\Delta} \right)
\]

7 This result follows from the need to retain the same angle of view and the properties of similar triangles.

8 The scaling ratio \( \rho \) is

\[
\rho = \frac{\Delta_1}{\Delta_2} = \frac{\tan \frac{\alpha_2}{2}}{\tan \frac{\alpha_1}{2}}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are the corresponding fields of view.

9 In his book Secret Knowledge [7] the artist David Hockney and the physicist Charles Falco suggest that many artists used the optical approach. They claim to be able to detect perspectival errors that can best be attributed to optical (rather than geometrical construction or mechanical construction) errors.

10 A Canon 100mm macro lens was used with the smallest aperture to allow focusing on the close and oblique surface of the computer monitor.

11 This so called anamorphic projection is discussed by Kemp [2], pp. 208–209 and Hockney [7], pp. 56–57

12 The image was rotated by 80° about the vertical axis and 35° about the horizontal axis using the map plug-in to GIMP – an image editor.

13 It is interesting that the left-hand ambassador in Fig. 8(b) appears to point at the skull with an elongated forefinger – actually the scroll appearing in Fig. 8(a).

14 According to Snyder [12], the Stereographic projection was known to Hipparchus and Ptolemy and is thus also an ancient projection.

15 From Paris, BnF, département des Manuscrits, Francais 6465, fol. 442 (Livre de Charles V). This forms Plate 55a of White [1], who discusses this in detail in Chapter XV.

16 This is plate 424 in Kemp [2] who suggests that is may be produced with the aid of a device of Lanci [2, pl.337]

17 As noted by Maillet [13]: “Pope John XXII .. in 1326 issued a Bull Super Illius Specula declaring that the devil is likely to be enclosed in mirrors and excommunicating those who attempt to practice catopromancy – divination with mirrors”.

References


Figure 2: Linear Perspective. The left-hand column is a computer-generated tiled floor and the right-hand column is a view of Loch Ossian. “FoV” is the horizontal field of view. If the image were 1m wide, the correct viewing distances would be 2.51m, 1.21m, 0.50m and 0.21m respectively.
Figure 3: Two examples of linear perspective. Comparing the tiles with those in Figure 2, a) appears to be wide angle (about $90^\circ$) and b) to be narrow angle (about $22.5^\circ$).
Figure 4: Creating perspective mechanically
Figure 5: Jan van der Heyden: The New Town Hall in Amsterdam, 1677
Figure 6: Jan van der Heyden: The New Town Hall in Amsterdam, 1677
Figure 7: Wide angle (108°) images by the author, 2009.
Figure 8: Holbein: The Ambassadors 1533
(a) Pearblossom Hwy

(b) Merced River, Yosemite: 1982

Figure 9: David Hockney: Photo collage
Figure 10: A baroque church ceiling in Prague
(a) Dürer: Star Map, Stereographic Projection.

(b) Leonardo da Vinci: Octant world map, 1514

Figure 11: Renaissance maps
Figure 12: Spherical Perspective.
(a) Jean Fouquet: Entree de l’Empereur Charles IV a Saint-Denis, 1460

(b) Fabritus, View of Delft, 1652

Figure 13: Cylindrical Perspective